Perfectoid Spaces

31.10.2024
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- A non-arch.
- $T_i \subseteq A$ subsets for $i \in I$ s.t.: $\overbrace{\forall i \in I\ \forall n \geq 0\ \forall O \in U \text{ open}: T_i^n \cdot U \text{ is open}}^{(*)}$
  (where $T_i^n \cdot U$ is the additive subgroup generated by all appropriate products of elements)

Let $s_i \in A$ for $i \in I$. Localization at $\left(\frac{T_i}{s_i} \mid i \in I\right)$ is denoted $A\left(\frac{T_i}{s_i} \mid i \in I\right)$.
It is a non-arch. ring, whose underlying ring is just $\{s_i \mid i \in I\}^{-1} A$.
A fund. syst. of top. is given by $(\underset{\text{ring generated by } 1}{\text{prime ring}})\left[\frac{t}{s_i} \mid i \in I, t \in T_i\right] \cdot U$ ← open additive subgroup of A. $=: U\left[\frac{T_i}{s_i} \mid i \in I\right]$

$\boxed{\text{Ex}}$: $R\left(\frac{f_1, \dots, f_n}{g}\right) \subseteq \mathrm{Spa}(A, A^+)$. The correct ring will be the completion of
$A\left(\frac{\{f_1, \dots, f_n\} = T_1}{g = s_1}\right)$.

$\boxed{\text{Ex}}$: $\mathrm{Spa}(\mathbb{Q}_p\langle x\rangle, \mathbb{Z}_p\langle x\rangle) \supseteq \mathbb{Z}_p$
$\left(\sum a_i x^i \longmapsto |\sum a_i \lambda^i|_p\right) \longleftarrow\!\shortmid \lambda$

$R\left(\frac{1}{x}\right) = \{ |\cdot| \in \mathrm{Spa}(\dots) \mid |1| \leq |x| \neq 0, ~~\dots~~ \}$

$R\left(\frac{1}{x}\right) \cap \mathbb{Z}_p = \{ \lambda \in \mathbb{Z}_p \mid \overbrace{|1|_p}^{=1} \leq |\lambda|_p \} = \mathbb{Z}_p \setminus p\mathbb{Z}_p$

If there's truth in the universe, then we should have $x - \lambda \in \mathbb{Q}_p\langle x\rangle\langle \frac{1}{x}\rangle^\times$.

Topology on $(\mathbb{Q}_p\langle x\rangle)(\frac{1}{x})$? $p\mathbb{Z}_p\langle x\rangle[\frac{1}{x}] \in$ fund. system.

$\Rightarrow$ for every $\lambda \in p\mathbb{Z}_p$: $\sum_{i=0}^{\infty} \frac{\lambda^i}{x^{i+1}}$ Cauchy $\rightarrow$ In completion, it is $= \frac{1}{x - \lambda}$.

$\boxed{\text{Def}}$: A f-adic, (A_0, I) couple of definition. $\Rightarrow \hat{A} = \underset{\longleftarrow n}{\lim} A/I^n$ ← as abelian groups

↳ a priori not a ring!

$\boxed{\text{Ex}}$: $A_0 = \mathbb{Z}_p\langle x\rangle[\frac{1}{x}]$, $A = \mathbb{Q}_p\langle x\rangle[\frac{1}{x}]$, $I = p \cdot A_0$

$\boxed{\text{Prop}}$: $\hat{A}$ carries a unique top. ring structure s.t. $\begin{aligned} A &\longrightarrow \hat{A} \\ a &\longmapsto (a \bmod I^n) \end{aligned}$ is a continuous ring homomorphism.

Multiplication: $(x_n) \cdot (y_n) = (\underset{\text{lift to } A}{\overline{x_n} \cdot \overline{y_n}} \bmod I^n)$.

$\boxed{\text{Key}}$: It is well-defined modulo some smaller powers of I because
$\exists m > 0$ s.t. $I^m \cdot \bar{x}_1 \subseteq I$ (works for every lift $\Rightarrow$ works for every $\bar{x}_n$).

$\boxed{\text{Def}}$: $f: A \to B$ continuous ring hom. between affine rings. f is said to be <u>adic</u> if for any couples of def (A_0, I), (B_0, J), we have $f(A_0) \subseteq B_0$ and $f(I) \cdot B_0$ an ideal of definition. <u>Missing</u>: indep. of choice of couples of def

$\boxed{\text{Ex}}$:

$$\begin{array}{ccc} \mathbb{Q}_p & \hookrightarrow & \mathbb{Q}_p\langle x \rangle \\ \cup\!\! \mid & & \cup\!\! \mid \\ \mathbb{Z}_p & \hookrightarrow & \mathbb{Z}_p\langle x \rangle \end{array}$$

$\boxed{\text{Rem}}$: Completion is adic, and the completion is f-adic

$\hookrightarrow$ <u>Need</u>: for $(a_n) \in \varprojlim I/I^n \Rightarrow (a_n) = \sum_{\text{finite}} b_i \cdot \overbrace{(c_n)}^{\in \hat{A}}$

<u>Key</u>: M ~~finitely generated~~ module over A_0, then $I \cdot \hat{M} = \ker(\hat{M} \to M/IM)$
(completion of $M^{\oplus s} \twoheadrightarrow I.M$)

$\boxed{\text{Rem}}$: Also, $\hat{A} = \hat{A}_0 \otimes_{A_0} A$.

$\boxed{\text{Ex}}$ $(\mathbb{Q}_p\langle x \rangle)\left(\frac{1}{x}\right) := \widehat{(\mathbb{Q}_p\langle x \rangle)\left(\frac{1}{x}\right)}$

$\|?$

$\varprojlim \left(\mathbb{Z}_p\langle x \rangle [\tfrac{1}{x}] / p^n \ldots \right) [\tfrac{1}{p}]$

$\boxed{\text{Last bit}}$: localization of f-adic is f-adic:
Let (A_0, I) be a couple of def of A.

$\boxed{\text{Lemma}}$: $T_i \cdot A$ open $\Rightarrow$ $(*)$ is satisfied.

$\boxed{\text{Pf}}$: wma $U = I^m$, $T_i^n U = T_i^n \cdot I^n = T_i^n \cdot A \cdot I^n = I^{n+m}$ $\square$

Instead of $(*)$: $T_i \cdot A$ open ideal $\forall i \in I$.

$A\left(\frac{T_i}{s_i} \mid i \in I\right) \supseteq A_0\left[\frac{T_i}{s_i} \mid i \in I\right] \left(=: \overbrace{(\text{prime ring})\left[\tfrac{t}{s_i} \mid i \in I, t \in T_i\right]}^{R} \cdot A_0\right)$.

by def, $I^n\left[\frac{T_i}{s_i} \mid i \in I\right]$ gives the topology $\Rightarrow$ f-adic.
"$\left(I\left[\frac{T_i}{s_i} \mid i \in I\right]\right)^n$"

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Assume $I = \{1\}$. In this setting, topology on $A[x] \rightsquigarrow A[x]_T$.

The open subsets are given by: take $U \subseteq A$ open subgroup

$\hookrightarrow U_{[x,T]} := \{ f = \sum a_i x^i \in A[x] \mid a_i \in T^i U \}$.

Define also $A\langle x \rangle_T \subseteq A[\![x]\!]$ by:

$A\langle x \rangle_T = \{ f = \sum a_i x^i \mid \forall\, 0 \in U$ open subgroup, $a_i \in U$ with finitely many exceptions$\}$.

$U_{\langle x,T \rangle} := \{ f = \sum a_i x^i \mid \forall i : a_i \in T^i U \}$.

$\boxed{\text{Ex}}$: $\mathbb{Q}_p\langle x \rangle = \mathbb{Q}_p\langle x \rangle_T$ for $T = \{1\}$

$\boxed{\text{Rem}}$: $(\mathbb{Q}_p\langle x \rangle)\langle v \rangle_{\{p,x\}} \twoheadrightarrow (\mathbb{Q}_p\langle x \rangle)\left\langle \frac{p, x}{p} \right\rangle$

$\ker = (vp - 1) \qquad v \longmapsto 1/p \qquad \rightsquigarrow$ with quotient topology.